# Homework 2 (Due 02/05/2014)

Math 622

January 31, 2014

**Change: Problem 4 (vi) fixed typos, delete $\sigma_1$ in the first equation, changed $dM(t)$ in the second equation to $dN_1(t)$**

1. Let $N$ be a Poisson process with rate $\lambda$ and filtration $\mathcal{F}(t)$. Define

$$Y(t) := \exp\Big(uN(t) - \lambda t(e^u - 1)\Big),$$

that is $Y$ is the exponential martingale associated with $N$. Use stochastic calculus (Ito's formula) for jump processes to show that

$$Y(t) = 1 + \int_0^t (e^u - 1)Y(s-)dM(s),$$

where $M(t) = N(t) - \lambda t$ and conclude that $Y(t)$ is a martingale w.r.t $\mathcal{F}(t)$.

2. Let $N_1(t), N_2(t)$ be independent Poisson processes with rate $\lambda_1, \lambda_2$ and $\mathcal{F}(t)$ a filtration for both $N_1, N_2$. Also define $M_i(t) = N_i(t) - \lambda_i t, i = 1, 2$.

(i) Show that the probability that $N_1$ and $N_2$ have the same jump time is 0 (Hint: Apply two dimensional Ito's formula for processes with jumps to $M_1(t)M_2(t)$ and take expectations on both sides).

(ii) Let

$$Y(t) = \exp\big(u_1N_1(t) + u_2N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1)\big).$$

Use a similar technique like problem 1 (i.e do not use direct computation) to show hat $Y(t)$ is a martingale with respect to $\mathcal{F}(t)$. (Part (i) may also be helpful here).

3. The converse of 2 (ii) is also true: if $Y(t)$ is a martingale for any $u_1, u_2$ then $N_1, N_2$ are independent Poisson processes with rate $\lambda_1, \lambda_2$. You may use this fact in this problem.

On a probability space $(\Omega, \mathbb{P})$ let $N_1, N_2$ be independent Poisson processes with rate $\lambda_1, \lambda_2$ and $\mathcal{F}(t)$ a filtration for $N_1, N_2$. Fix $a_1, a_2$. Define the probability $\mathbb{Q}$ by

$$\begin{aligned} Z(T) &:= \exp\Big[a_1 N_1(T) + a_2 N_2(T) - \lambda_1 T(e^{a_1} - 1) - \lambda_2 T(e^{a_2} - 1)\Big] \\ d\mathbb{Q} &= Z(T) d\mathbb{P}. \end{aligned}$$

(i) Show that under $\mathbb{Q}$, $N_1, N_2$ are independent Poisson processes with rates $\lambda_1 e^{a_1}, \lambda_2 e^{a_2}$ respectively.

(ii) Find $Z(T)$ so that if we define $d\mathbb{Q} = Z(T)d\mathbb{P}$ then $N_1, N_2$ are independent Poisson processes with rates $\tilde{a}_1, \tilde{a}_2$ respectively.

4. On a probability space $(\Omega, \mathbb{P})$ let $N_1, N_2$ be independent Poisson processes with rate $\lambda_1, \lambda_2$ and $\mathcal{F}(t)$ a filtration for $N_1, N_2$. Assume $b_1 > 0 > b_2 > -1$ and let

$$Q(t) := b_1 N_1(t) + b_2 N_2(t).$$

(i) Find $m$ so that $M(t) = Q(t) - mt$ is a $\mathcal{F}(t)$-martingale.

(ii) Consider the price model

$$dS(t) = \alpha S(t)dt + S(t-)dM(t), S(0) = 1.$$

Write down a solution in the form $S(t) = Ke^{a_0 t + a_1 N_1(t) + a_2 N_2(t)}$; identify the constants $a_0, a_1, a_2$ and $K$.

(iii) Let $\alpha = r$, where $r$ is the risk free interest rate, for the model in part (ii), then the measure $\mathbb{P}$ is risk-neutral. So the price of a Euro-call option that pays $V(T) = (S(T) - K)^+$ at time $T$ is

$$V(t) = e^{-r(T-t)}\mathbb{E}[(S(T) - K)^+ | \mathcal{F}(t)].$$

Find an explicit formular for $V(t)$ in the style of the formula (11.7.3) on page 507 of Shreve. Your final answer will be a doubly infinite sum.

(iv) Suppose now that $a \neq r$ for the model in part (ii). Show how we can define a risk-neutral measure $\mathbb{Q}$ for the model in (ii) (Hint: use the result in 3 (ii)).

(v) Show that there are in fact many different risk-neutral measures for the setting in (iv).

(vi) Suppose that now we consider a market with 2 assets:

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t)dt + S_1(t-)dM(t), S_1(0) = 1 \\ dS_2(t) &= \alpha_2 S_2(t)dt + \sigma_2 S_2(t-)dN_1(t), S_2(0) = 1, \end{aligned}$$

where $\sigma_2 > 0$. A risk neural probability $\mathbb{Q}$ for this market must be such that $e^{-rt}S_1(t)$ and $e^{-rt}S_2(t)$ are $\mathcal{F}(t)$ martingales under $\mathbb{Q}$. Show how we can define a risk neutral measure $\mathbb{Q}$ for this market. What conditions must $\alpha_1, \alpha_2, \sigma_1, \sigma_2, \lambda_1, \lambda_2, r$ satisfy for this measure change to be valid? When is $\mathbb{Q}$ unique? (It is helpful to look at the discussion in Shreve's pagge 516).

5. Let $T_i, i = 1, ..., k$ be independent exponentially distributed random variables with rate $\lambda_i, i = 1, ..., k$.

(i) Let $U = \min_{i=1,...,k} T_i$ and $V = \max_{i=1,...,k} T_i$ . Find the density functions of $U$ and $V$.

(ii) Show that $P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

6.

(i) Suppose $S_t$ satisfies

$$S(t) = 1 + \int_0^t \alpha S(u)du + \sum_{0<u\leq t} \Delta J(u),$$

where $J(u)$ is a pure jump function. Solve for an explicit formula for $S(t)$.

Explanation: So far we've studied the model of

$$\begin{aligned} S(t) &= 1 + \int_0^t \alpha S(u)du + \int_0^t S(u-)dJ(u) \\ &= 1 + \int_0^t \alpha S(u)du + \sum_{0<u\leq t} S(u-)\Delta J(u). \end{aligned}$$

It is natural to ask how the solution changes if the term $S(u-)$ disappears in the equation. There are 2 ways to solve this question: a) let $0 < t_1 < t_2 < ...$ be the jump times of $J$. Solve for $S(t)$ on each interval $t_i < t < t_{i+1}$ (note the strict inequality) and consider what happens at each $t_i$. b) Note that we have a simpler way to write $\sum_{0<u\leq t} \Delta J(u)$. Apply Ito's formula to $e^{-\alpha t}S_t$ and see what happens.

(ii) Now suppose $S_t$ satisfies

$$S(t) = 1 + \int_0^t \alpha(u)S(u)du + \sum_{0<u\leq t} \sigma\Delta J(u).$$

Solve for an explicit formula for $S(t)$.